Arithmetic Groups Acting on Products of Trees

Group Actions and Dynamics (GAGTA 2026)

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This presentation contains joint work with Jakob Stix, Alina Vdovina, and Jiahui Yu

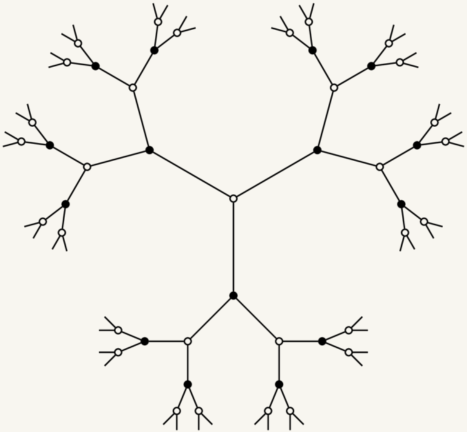
Bruhat–Tits Trees

For $k \geq 2$, let $\mathcal{T}_k$ be a k-regular tree.

Recall that if K is a non-archimedean local field, i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, with residue field of size q, then there is a natural transitive action

$$\mathrm{PGL}_2(K) \curvearrowright \mathcal{T}_{q+1},$$

where $\mathcal{T}_{q+1}$ is the Bruhat-Tits tree.



Euler Characteristic

For a fixed non-archimedean field K, we are interested in finding explicit irreducible, torsion-free lattices

$$\Lambda \leq \prod_{i=1}^{d} \mathrm{PGL}_2(K), \quad d \geq 2,$$

with small Euler characteristic. Note that all such lattices are arithmetic by Margulis. We see that

$$\Lambda \curvearrowright \mathcal{T}_{q+1}^d, \quad \text{and} \quad \mathcal{X}_\Lambda = \Lambda \backslash \mathcal{T}_{q+1}^d$$

is a CAT(0) cube complex. If M_Λ is the number of vertex orbits, then

$$\chi(\Lambda) = \chi(\mathcal{X}_\Lambda) = M_\Lambda \cdot \left(\frac{1-q}{2}\right)^d .$$

Prior Work

The first explicit examples of such lattices were constructed by Stix and Vdovina.

Theorem (Stix–Vdovina, 2017)

For all odd prime powers q, there exists an explicit, torsion–free, irreducible lattice

$$\Lambda \leq \mathrm{PGL}_2(\mathbb{F}_q((t))) \times \mathrm{PGL}_2(\mathbb{F}_q((t)))$$

with $M_\Lambda = 1$. In particular, if $q = 3$ then $\chi(\Lambda) = 1$.

Note that each Λ yields a 1-vertex square complex. We emphasize that their computations are explicit, meaning they can write down the square complex

$$\mathcal{X}_\Lambda = \Lambda \backslash \mathcal{T}_{q+1} \times \mathcal{T}_{q+1}.$$

Prior Work

Vdovina also constructed a torsion-free lattice Λ acting on $\mathcal{T}_3 \times \mathcal{T}_3$ with four vertex orbits. She conjectured that Λ is arithmetic; this was proved by Rungtanapirom.

Theorem (Rungtanapirom, 2018)

There exists a torsion-free, irreducible lattice

$$\Lambda \leq \mathrm{PGL}_2(\mathbb{F}_2((t))) \times \mathrm{PGL}_2(\mathbb{F}_2((t)))$$

with $M_\Lambda = 4$*. In particular,* $\chi(\Lambda) = 1$*.*

If $d = 2$, then $\chi(\Lambda) = 1$ if and only if $(q, M_\Lambda) = (2, 4)$ or $(3, 1)$.

Prior Work

Rungtanapirom, Stix, and Vdovina extend their constructions to rank ≥ 3. We give a simplified formulation of their work.

Theorem (Rungtanapirom–Stix–Vdovina, 2019)

If $d \geq 1$, then for every odd prime power $q > d+1$, there exists a torsion-free, irreducible lattice

$$\Lambda \leq \prod_{i=1}^{d} \mathrm{PGL}_2(\mathbb{F}_q((t)))$$

with $M_\Lambda = 1$. The presentation of Λ can also be reasonably computed in Magma.

For instance, they construct an explicit, torsion–free, irreducible lattice

$$\Lambda \leq \mathrm{PGL}_2(\mathbb{F}_5((t))) \times \mathrm{PGL}_2(\mathbb{F}_5((t))) \times \mathrm{PGL}_2(\mathbb{F}_5((t)))$$

with $M_\Lambda = 1$.

Main Results

We now show similarities and differences in the p-adic setting.

Theorem A (Stix–M.–Vdovina)

If

$$\Lambda \leq \mathrm{PGL}_2(\mathbb{Q}_3) \times \mathrm{PGL}_2(\mathbb{Q}_3),$$

is a torsion-free, cocompact, irreducible lattice, then $M_\Lambda \geq 2$, and there is a unique lattice with $M_\Lambda = 2$.

Furthermore, if

$$\Lambda \leq \mathrm{PGL}_2(\mathbb{Q}_2) \times \mathrm{PGL}_2(\mathbb{Q}_2)$$

is a torsion-free, cocompact, irreducible lattice, then $M_\Lambda \geq 12$, and there exists a lattice with $M_\Lambda = 12$.

Theorem B (Stix–M.–Vdovina)

If $p = 5, 7$ or $p \equiv \pm 1 \bmod 5$, then there exists a torsion–free, irreducible lattice

$$\Lambda \leq \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_p),$$

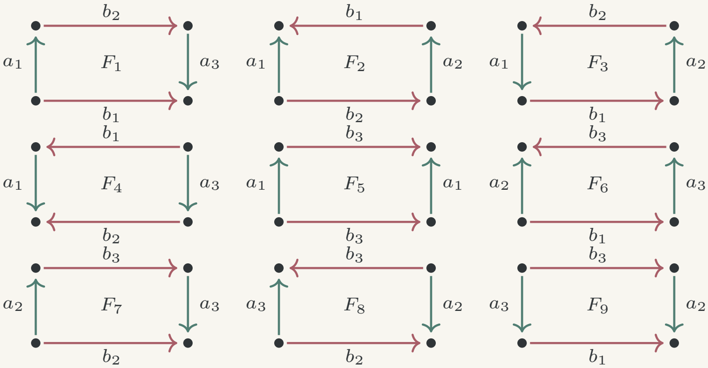
with $M_\Lambda = 1$.

For $p = 5$, one example is

$$\Lambda = \left\langle a_1, a_2, a_3, b_1, b_2, b_3 \;\middle|\; \begin{array}{lll} a_1 b_2 a_3 b_1^{-1}, & a_1 b_1^{-1} a_2^{-1} b_2^{-1}, & a_1^{-1} b_2^{-1} a_2^{-1} b_1^{-1}, \\ a_1^{-1} b_1^{-1} a_3 b_2, & [a_1, b_3], & a_2 b_3^{-1} a_3^{-1} b_1^{-1}, \\ a_2 b_3 a_3 b_2^{-1}, & a_3 b_3^{-1} a_2 b_2^{-1}, & a_3^{-1} b_3 a_2 b_1^{-1} \end{array} \right\rangle.$$

The $p = 5$ Squares

We now describe the square complex $\mathcal{X}_\Lambda$ for the $p = 5$ example. The link of its unique vertex is the complete bipartite graph $K_{6,6}$. It follows that $\mathcal{X}_\Lambda$ has exactly 9 squares.



Main Results

What about $d = 3$?

Theorem C (Stix–M.–Vdovina)

If $p \equiv \pm 1 \bmod 13$, then there exists a torsion-free, irreducible lattice

$$\Lambda \leq \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_p),$$

with $M_\Lambda = 1$.

Outline for the rest of the talk

1. Background on number fields and quaternion algebras
2. Minimal lattices in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_p)$
3. Higher-dimensional lattices
4. Classification of Euler characteristic 1 lattices (if time permits)

The common theme is that these arithmetic lattices are explicit. That is, we know

- arithmetic information,
- group presentations,
- square and cube complexes.

Number fields and S-integers

A number field K is a finite extension of $\mathbb{Q}$. The equivalence classes of absolute values on K are called the places of K. A place is either archimedean or non-archimedean. Let K_v is the completion of K at v.

Let $V_f(K)$ be the set of non-archimedean places of K. Given a finite set $S \subseteq V_f(K)$, the S-integers of K are

$$\mathfrak{o}_{K,S} := \{x \in K : v(x) \geq 0 \text{ for all finite places } v \notin S\}.$$

If $S = \varnothing$, then $\mathfrak{o}_K = \mathfrak{o}_{K,\varnothing}$ is the usual ring of integers, i.e. the integral closure of $\mathbb{Z}$ in K.

There is a 1–1 correspondence between $V_f(K)$ and the prime ideals of $\mathfrak{o}_K$. We say that a prime integer p splits in K if

$$p\mathfrak{o}_K = \mathfrak{p}_1 \cdots \mathfrak{p}_n, \quad n = [K : \mathbb{Q}], \quad K_{\mathfrak{p}_i} \cong \mathbb{Q}_p \text{ for each } i.$$

For a split prime p, let S_p be the primes “above” p.

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For a split prime p, let S_p be the primes “above” p.

Quaternion Algebras

Let K be a field. A quaternion algebra over K is a central simple K-algebra of dimension 4.

When $\mathrm{char}(K) \neq 2$, every quaternion algebra has a presentation

$$B = \left(\frac{a, b}{K}\right) := K + Ki + Kj + Kij$$

with $a, b \in K^\times$ and

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji.$$

There are 2 possibilities: $B \cong \mathrm{M}_2(K)$ or B is a division algebra (Wedderburn's Theorem).

Standard references are Maclachlan-Reid, Vignéras, and Voight.

Orders in Global Fields

Let

- K be a number field,
- $S \subseteq V_f(K)$ be a finite set of places
- B be a quaternion algebra over K.

An $\mathfrak{o}_{K,S}$-order $\mathfrak{O}$ in B is a subring $\mathfrak{O} \subset B$ which is also a finitely generated $\mathfrak{o}_{K,S}$-module with

$$\mathfrak{O} \otimes_{\mathfrak{o}_{K,S}} K = B.$$

We alway take an order to be maximal: not properly contained in another $\mathfrak{o}_{K,S}$-order.

Splitting and Ramification

For a place v of K, there is an embedding $\sigma : K \to K_v$ and we set

$$B_v := B \otimes_K K_v = \left(\frac{\sigma(a), \sigma(b)}{K_v}\right).$$

We say:

- B splits at v if $B_v \cong M_2(K_v)$.
- B ramifies at v if B_v is a division algebra.
- B is totally definite if B_v ramifies at each archimedean place of K. This forces K to be totally real.

We always assume that B is totally definite.

Normalizer and Unit Groups

Suppose that $\mathfrak{O} \subseteq B$ is an $\mathfrak{o}_K$-order, and S only contains finite places at which B splits. Then

$$\mathfrak{O}_S = \mathfrak{O} \otimes_{\mathfrak{o}_K} \mathfrak{o}_{K,S}$$

is an $\mathfrak{o}_{K,S}$-order. We define the normalizer and unit group

$$\Gamma_{\mathfrak{O},S} = N_{B^\times}\left(\mathfrak{O}_S^\times\right)/K^\times \quad \supseteq \quad \Delta_{\mathfrak{O},S} = \mathfrak{O}_S^\times/\mathfrak{o}_{K,S}^\times.$$

Since B splits at each prime in S, there is an embedding

$$B^\times/K^\times \longrightarrow \prod_{\mathfrak{p}\in S} \mathrm{PGL}_2(K_\mathfrak{p}).$$

Because B is totally definite, $\Gamma_{\mathfrak{O},S}$ and $\Delta_{\mathfrak{O},S}$ are cocompact lattices under this embedding by Borel and Harish-Chandra.

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Computing Presentations

In Magma, together with Stix and Vdovina, we implement an algorithm for computing a $\Gamma_{\mathfrak{O},S}$-fundamental domain

$$\mathcal{F} \subseteq \mathcal{T}_S.$$

Our algorithm relies crucially on the work of Kirschmer and Voight. From $\mathcal{F}$, we can write down a presentation of $\Gamma_{\mathfrak{O},S}$ following Brown.

With a little more effort, we can describe the quotient $\Gamma_{\mathfrak{O},S} \backslash \mathcal{T}_S$ as a complex of groups and compute a presentation following Bridson and Haefliger.

The complex–of–groups viewpoint records cell stabilizers in every dimension, which is useful when searching for torsion-free lattices.

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The complex–of–groups viewpoint records cell stabilizers in every dimension, which is useful when searching for torsion-free lattices.

A small lattice in $\mathrm{PGL}_2(\mathbb{Q}_3) \times \mathrm{PGL}_2(\mathbb{Q}_3)$

If $K = \mathbb{Q}(\sqrt{13})$, then 3 splits in K, so we take $S = S_3$. Let

$$B = \left(\frac{-1,-1}{K}\right).$$

If $\mathfrak{O}$ is a maximal order in B, then in Magma we compute

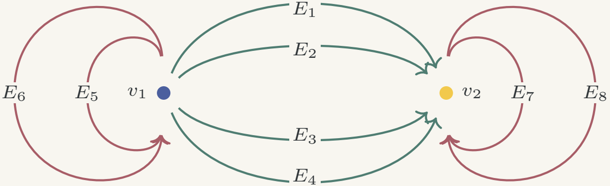
$$\Gamma_{\mathfrak{O},S} \cong \left\langle a,b,c \;\middle|\; \begin{array}{cc} a^2 = b^2 = c^2 = 1, & (cb)^6 = 1, \\ (abcbab)^2 = 1, & (bcbca)^3 = 1 \end{array} \right\rangle.$$

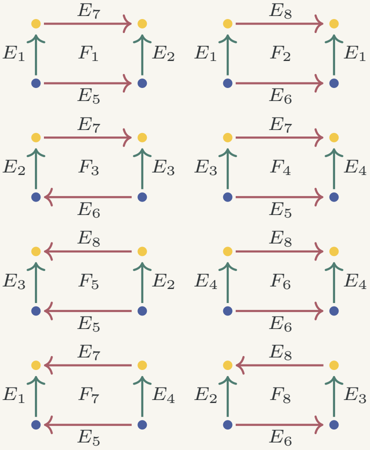
with representatives

$$\begin{aligned} a &= ij, \\ b &= 2i + (\sqrt{13} - 3)j + (1 - \sqrt{13})ij, \\ c &= 2i + (\sqrt{13} + 1)j - (\sqrt{13} + 3)ij. \end{aligned}$$

A small lattice in $\mathrm{PGL}_2(\mathbb{Q}_3) \times \mathrm{PGL}_2(\mathbb{Q}_3)$

There is a torsion-free subgroup $\Lambda_S \leq \Gamma_{\mathfrak{O},S}$ with $M_{\Lambda_S} = 2$. Thus, $\chi(\Lambda_S) = 2$.





Classification Theorem

We now give a stronger version of Theorem A.

Theorem (Stix–M.–Vdovina)

Let K be a number field and B/K a totally definite quaternion algebra with order $\mathfrak{O}$. Suppose that $S = \{\mathfrak{p}, \mathfrak{q}\}$ is a set of split primes with

$$K_{\mathfrak{p}} \cong K_{\mathfrak{q}}$$

and

$$\Lambda_S \leq \mathrm{PGL}_2(K_{\mathfrak{p}}) \times \mathrm{PGL}_2(K_{\mathfrak{q}})$$

is torsion–free, commensurable with $\Gamma_{\mathfrak{O},S}$ and has $\chi(\Lambda_S) \leq 2$. Then B, K, $\mathfrak{O}$, S, and Λ_S are as in the previous slide.

A small lattice in $\mathrm{PGL}_2(\mathbb{Q}_2) \times \mathrm{PGL}_2(\mathbb{Q}_2)$

If $K = \mathbb{Q}(\sqrt{33})$, then 2 splits in K, so we take $S = S_2$. Let

$$B = \left(\frac{-1, (\sqrt{33} - 7)/2}{K} \right).$$

If $\mathfrak{O}$ is a maximal order in B, then in Magma we compute

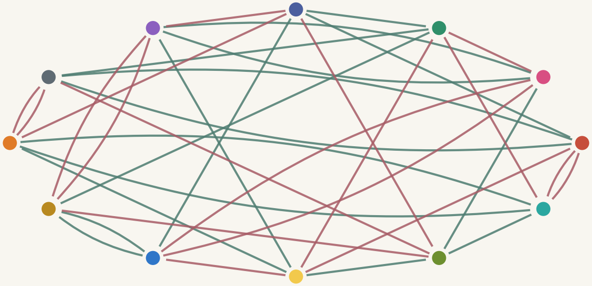
$$\Gamma_{\mathfrak{O},S} \cong \left\langle a, b, c \;\middle|\; \begin{matrix} a^2 = c^2 = (ac)^3 = 1, & ab^{-1}cacb^{-1}ac = 1, \\ (ab^{-1}ab)^2 = 1, & (b^{-1}ab^3ab^{-1})^2 = 1, \\ (b^2ab^{-2}ab)^2 = 1, & (abab^{-2}abab)^2 = 1 \end{matrix} \right\rangle.$$

with representatives

$$\begin{aligned} a &= ij, \\ b &= 12 - 12i + (\sqrt{33} + 3)j - (\sqrt{33} + 3)ij, \\ c &= (4\sqrt{33} + 24)i + (5\sqrt{33} + 27)j + (4\sqrt{33} + 24)ij. \end{aligned}$$

A small lattice in $\mathrm{PGL}_2(\mathbb{Q}_2) \times \mathrm{PGL}_2(\mathbb{Q}_2)$

There is a torsion-free subgroup $\Lambda_S \leq \Gamma_{\mathfrak{O},S}$ with $M_{\Lambda_S} = 12$. Thus, $\chi(\Lambda_S) = 3$. We conjecture that this example is the unique minimal Euler characteristic lattice in $\mathrm{PGL}_2(\mathbb{Q}_2) \times \mathrm{PGL}_2(\mathbb{Q}_2)$. We give the 1-skeleton below.



Infinite Families of Arithmetic 1-Vertex Square Complexes

Let $K = \mathbb{Q}(\sqrt{5})$. If a prime $p \equiv \pm 1 \mod 5$ then p splits in K. Let $S = S_p$. If $\mathfrak{O}$ is a maximal order in

$$B = \left(\frac{-1, -1}{K} \right),$$

then $\Delta_{\mathfrak{O},S}$ acts transitively on $\mathcal{T}_S$, with vertex stabilizers conjugate to

$$\mathfrak{O}^\times / \mathfrak{o}_K^\times \cong A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5).$$

In Magma, we find that

$$\pi_{\mathfrak{p}_5} : \Delta_{\mathfrak{O},S} \to \mathrm{PGL}_2(\mathbb{F}_5)$$

restricts to an injection on $\mathfrak{O}^\times / \mathfrak{o}_K^\times$. Thus, if $H \leq \mathrm{PGL}_2(\mathbb{F}_5)$ has order 2 then

$$\Lambda_S = \pi_{\mathfrak{p}_5}^{-1}(H),$$

acts simply transitively on the vertices of $\mathcal{T}_S$.

Theorem (Stix–M.–Vdovina)

Λ_S is torsion–free. In particular, Theorem A holds.

Proof Sketch.

If $[\lambda] \in \Lambda_S$ has finite order, then $[\lambda]$ is 2-torsion. Hence $[\lambda]$ is represented by an element $\lambda \in \mathfrak{O}_S^\times$ with reduced trace zero. The key computation is that

$$\pi_{\mathfrak{p}_5}([\lambda]) \in H \text{ and } \operatorname{trd}(\lambda) = 0 \quad \Longrightarrow \quad \operatorname{Nrd}(\lambda) \bmod \mathfrak{p}_5 \notin \mathbb{F}_5^{\times 2}.$$

The generators of Λ_S may be represented by $\alpha \in \mathfrak{O}_S^\times$ with

$$\operatorname{Nrd}(\alpha)\mathfrak{o}_K = \mathfrak{p}$$

for some $\mathfrak{p} \in S$. Since $N\mathfrak{p} \equiv \pm 1 \bmod 5$, a generator of $\mathfrak{p}$ modulo $\mathfrak{p}_5$ is in $\mathbb{F}_5^{\times 2}$. Thus, $\pi([\lambda]) \in \mathbb{F}_5^{\times 2}$ for all $[\lambda] \in \Lambda_S$. □

Higher-Dimensional Number Field Examples

We now prove Theorem C. Consider

$$K = \mathbb{Q}(t), \quad t^3 - t^2 - 4t - 1 = 0,$$

which is a degree 3, abelian, totally real subfield of $\mathbb{Q}(\zeta_{13})$. It follows from class field theory that a prime p totally splits in K if and only if

$$p \equiv \pm 1, \pm 5 \bmod 13.$$

Let $S = S_p$ and $\mathfrak{O}$ be a maximal order in

$$B = \left(\frac{-7t^2 - 14t - 9, -t - 2}{K} \right).$$

Then $\Delta_{\mathfrak{O},S}$ acts simply transitively on the vertices of $\mathcal{T}_S$.

Theorem (Stix–M.–Vdovina)

$\Delta_{\mathfrak{O},S}$ *is torsion-free whenever* $p \equiv \pm 1 \bmod 13$.

Another 3-Dimensional Example Covered by 6-Regular Trees

Let K and B be as in the previous slide. Then 5 totally splits but $\Delta_{\mathfrak{O},S}$ is not torsion-free if $S = S_5$.

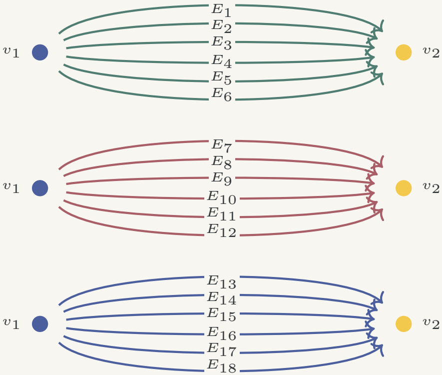
However, there exists a torsion–free index 2 subgroup $\Lambda_S \leq \Delta_{\mathfrak{O},S}$ with

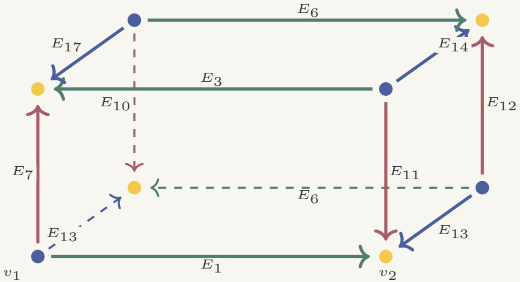
$$M_{\Lambda_S} = 2 \quad \text{and} \quad \chi(\Lambda_S) = -16.$$

We conjecture that Λ_S has minimal $|\chi|$ among all arithmetic cube complexes of dimension at least 3 that are defined over number fields and covered by products of trees of constant regularity.

Another 3-Dimensional Example Covered by 6-Regular Trees

We illustrate the 1-skeleton of $\mathcal{X}_{\Lambda_S}$, and give one example of a 3-cube in $\mathcal{X}_{\Lambda_S}$.





Simply Transitive Actions

It is an interesting to ask when there is a subgroup $\Lambda \leq \Gamma_{\mathfrak{O},S}$ acting simply transitively on the vertices of $\mathcal{T}_S$.

Kirchmer and Voight classify all class number 1 maximal $\mathfrak{o}_K$-orders in quaternion algebras defined over a number field. Going case-by-case, we prove the following, partially answering a question of Lubotzky.

Theorem (M.–Yu)

Let K be a number field, and let B/K be a totally definite quaternion algebra with class number 1 maximal $\mathfrak{o}_K$-order $\mathfrak{O}$. Then there is a finite set of primes Ω such that, for every finite subset of split primes

$$S \subseteq V_f(K) \setminus \Omega,$$

the group $\Gamma_{\mathfrak{O},S}$ contains a congruence subgroup Λ_S acting simply transitively on the vertices of $\mathcal{T}_S$.

Theorem (Stix–M.–Vdovina)

Let K be a number field, B/K a totally definite quaternion algebra with maximal order $\mathfrak{O}$, and $S = \{\mathfrak{p}, \mathfrak{q}\}$ a finite set of split primes . Suppose that

$$\Lambda_S \leq \mathrm{PGL}_2(K_\mathfrak{p}) \times \mathrm{PGL}_2(K_\mathfrak{q})$$

is commensurable with $\Gamma_{\mathfrak{O},S}$ and has $\chi(\Lambda_S) = 1$. Then K is either

$$\mathbb{Q} \quad \text{or} \quad \mathbb{Q}(\sqrt{3}),$$

$\mathfrak{O}$ is a maximal order in

$$B = \left(\frac{-2,-5}{\mathbb{Q}}\right), \quad \left(\frac{-2,-13}{\mathbb{Q}}\right), \quad \text{or} \quad \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right),$$

and $S = \{\mathfrak{p}_2, \mathfrak{p}_3\}$. In each case, $\Lambda_S \leq \Gamma_{\mathfrak{O},S}$ and Λ_S is unique up to automorphism of $\Gamma_{\mathfrak{O},S}$.

Let $K = \mathbb{Q}$ and

$$B = \left(\frac{-2, -5}{\mathbb{Q}}\right),$$

which ramifies at 5. Take $S = \{2, 3\}$. If $\mathfrak{O}$ is a maximal order of B then in Magma we can compute

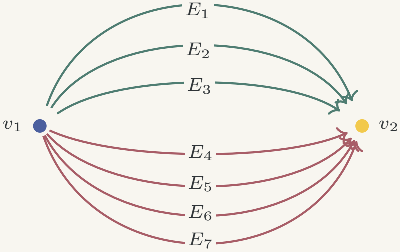
$$\Gamma_{\mathfrak{O},S} \cong \left\langle a, b, c, d \;\middle|\; \begin{array}{l} a^2 = b^2 = c^2 = d^2 = 1, \\ (ad)^2 = (bd)^2 = (ca)^2 = 1, \\ (ba)^3 = (bcadcbc)^2 = 1 \end{array} \right\rangle.$$

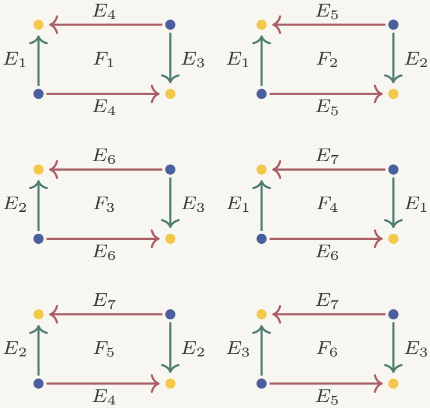
with

$$a = 5i - 2j + ij, \quad b = 5i + 2j + ij,$$
$$c = i + 2j + ij, \quad d = i - ij.$$

An Euler Characteristic 1 Square Complex

There is a torsion-free congruence subgroup $\Lambda_S \leq \Gamma_{\mathfrak{O},S}$ of index 12 with $\chi(\Lambda_S) = 1$.





Green edges represent $p = 2$ and red edges represent $p = 3$

Another Euler Characteristic 1 Square Complex

Let $K = \mathbb{Q}(\sqrt{3})$ and $\mathfrak{O}$ be a maximal order in the quaternion algebra

$$B = \left(\frac{-1,-1}{K}\right).$$

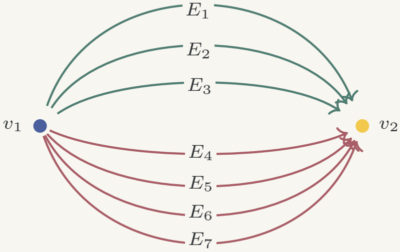
Note that 2 and 3 both ramify in K. Take $S = \{\mathfrak{p}_2, \mathfrak{p}_3\}$.

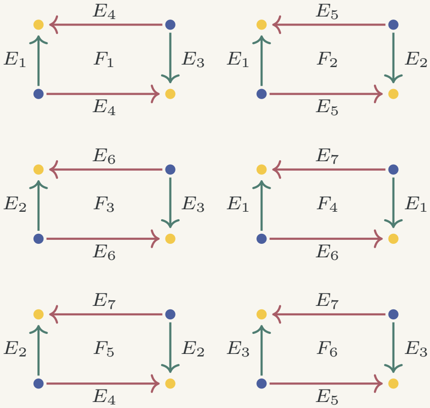
Then $\mathcal{X}_{\Gamma_{\mathfrak{O},S}}$ has 2 vertices and

$$\Gamma_{\mathfrak{O},S} \cong \left\langle a,b,c \;\middle|\; \begin{array}{l} a^2 = b^2 = (ab)^4 = (acbc^{-1})^2 = 1, \\ (c^{-1}bcbc^{-1})^2 = (abc^{-1}bc)^2 = (bc^{-1}bcbc^{-1})^2 = 1, \\ cbc^{-1}ababcbc^{-1}bab = 1 \end{array} \right\rangle.$$

Another Euler Characteristic 1 Square Complex

There exists a torsion-free congruence subgroup $\Lambda_S \leq \Gamma_{\mathfrak{O},S}$ with $\chi(\Lambda_S) = 1$.





Green edges represent $p = 2$ and red edges represent $p = 3$

Thank you!